A CLASS OF POLYTOPES WITH A REMARKABLE VOLUME FORMULA

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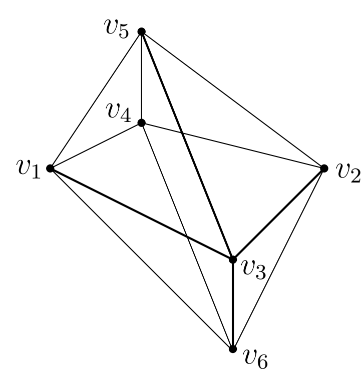
ABSTRACT. We say a polytope $P \in \mathbb{R}^d$ is "determinantal" if its volume can be expressed as a single $d \times d$ determinant whose columns are formed from the vertex vectors of P. In order to determine which polytopes are determinantal, we introduce a class of simplicial polytopes $\mathcal{E}^d$ which we call "endoskeletal". $\mathcal{E}^d$ is classified by the partitions $d = d_1 + \cdots + d_n$ of d, and the structure of a $P \in \mathcal{E}^d$ is based on a corresponding partition $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n$ of the vertices of P such that the simplices $B_j = \operatorname{conv} \mathcal{V}_j$, $j = 1, \cdots, n$, are internal to P. $\mathcal{E}^d$ includes a number of familiar polytopes, such as the octahedron in $\mathbb{R}^3$. In addition to proving that every $P \in \mathcal{E}^d$ is determinantal, we analyze the combinatorics of $\mathcal{E}^d$ and present various examples. We also construct related examples of determinantal polyhedra which are not endoskeletal.

KEY WORDS: polytopes, volume, simplex, determinant, partition
MSC: 52B11; 11C20, 52A38

1. Introduction and Definitions

Suppose you are given the coordinates of the vertices $v_1, \cdots, v_6$ of an octahedron O in $\mathbb{R}^3$. How would you calculate its volume?

Most likely you would triangulate O into simplices, use the standard determinant formula for the volume of each simplex (given in (2) below), and then add up these partial volumes to get $\operatorname{vol} O$. However, as we emphasized in our paper with William Kindree [4], there is a much better way! The volume of the pictured octahedron is actually given by a single determinant:



$$\operatorname{vol} O = \frac{1}{3!} \,|\det(v_1 - v_2, v_3 - v_4, v_5 - v_6)|, \tag{1}$$

where the columns of the 3×3 matrix in (1) are the three axis vectors.

(1) came as something of a surprise (to us) and led us to ask: What are the polytopes in $\mathbb{R}^d$ whose volume is given by a formula like (1) involving a single $d \times d$ determinant? We shall call such polytopes "determinantal". In this paper we introduce the class of "endoskeletal" polytopes which we regard as the natural answer to the above question. We must point out that while the vertices of an endoskeletal polytope like O may be placed in general positions, they are subject to a significant collective constraint (see Example 1 below).

We begin with some notation and standard terminology (see [3] for more details). For us, the word polytope will mean a bounded convex set $P \subset \mathbb{R}^d$ which is the convex hull of its set of vertices $\mathcal{V}(P)$. If a geometrical object has dimension m we call it an "m-object". The word hyperplane means a $(d-1)$-plane in $\mathbb{R}^d$. The hyperplane H is a supporting plane for the polytope P if H intersects the boundary ∂P of P but not its interior P°. A face of P is the intersection of P with a supporting plane. Let $\mathcal{F}_j(P)$ be the set of j-faces of P and let $f_j(P) = |\mathcal{F}_j(P)|$ be its jth face number; e.g., $f_0(P) = |\mathcal{V}(P)|$. The set of face numbers $\boldsymbol{f}(P) = (f_0(P), \cdots, f_{d-1}(P))$ is called the f-vector of P. The $(d-1)$-faces of P are called the facets of P.

A j-simplex is a j-polytope with the smallest possible $f_0(P)$, namely $f_0(P) = j+1$. Given a set X of $j+1$ convexly independent points, we denote the j-simplex with vertex set X by $S(X)$. Recall that the set $\mathcal{S}^d$ of simplicial polytopes in $\mathbb{R}^d$ consists of those d-polytopes all of whose facets are $(d-1)$-simplices. The volume of a d-simplex $S \subset \mathbb{R}^d$ with vertices $v_0, v_1, \cdots, v_d$ is [2]

$$\mathrm{vol}\, S = \frac{1}{d!} |\det(v_1 - v_0, \cdots, v_d - v_0)|. \tag{2}$$

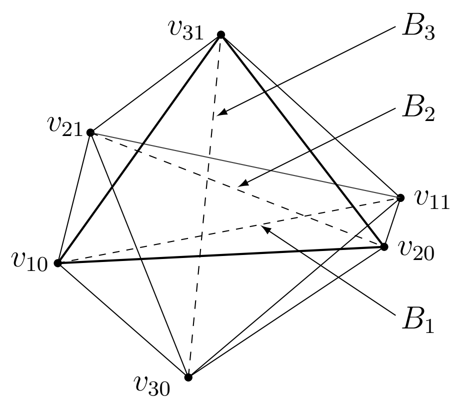


Figure 1. Octahedron O^3

To clarify our definition of the class of endoskeletal polytopes $\mathcal{E}^d \subset \mathcal{S}^d$, we revisit the prototypical example of an endoskeletal polytope, namely, the octahedron $O^3 \subset \mathbb{R}^3$, as displayed in Figure 1. O^3 has $f_0 = 6$ vertices, $f_1 = 12$ edges, and, as signified by its name, $f_2 = 8$ facets. There are 3 internal line segments or axes, marked by dashed lines in Figure 1: $B_1 = S(v_{10}, v_{11})$, $B_2 = S(v_{20}, v_{21})$, and $B_3 = S(v_{30}, v_{31})$. Note that the number of 2-simplices we can form using the vertices of O^3 is $\binom{6}{3} = 20$, and that the 8 facets are precisely those which do not contain a B_j.

We now describe a general polytope $P \in \mathcal{E}^d \subset \mathcal{S}^d$. The endoskeletal polytopes are classified by the partitions of d:

$$\pi: \quad d = d_1 + d_2 + \cdots + d_n, \tag{3}$$

where $d_1 \geq d_2 \geq ... \geq d_n \geq 1$. We define the class $\mathcal{E}^d_\pi$ of endoskeletal polytopes to be those polytopes in $\mathcal{S}^d$ whose vertex set $\mathcal{V}$ can be partitioned into a disjoint union,

$$\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n, \qquad |\, \mathcal{V}_j \,| = d_j + 1, \; j = 1, \cdots, n, \tag{4}$$

such that if $\mathcal{U}$ is a set of d vertices of P then $S(\mathcal{U})$ is a facet of P if and only if $\mathcal{U}$ contains no $\mathcal{V}_j$. We call this restriction the Facet Condition or FC. If $P \in \mathcal{E}^d_\pi$ we say that P is of Type $[\pi]$.

We call the simplices $B_j = S(\mathcal{V}_j)$, $j = 1, \cdots, n$, the bones of P and $\{B_1, \cdots, B_n\}$ its endoskeleton. Each vertex $v_{jk} \in \mathcal{V}(P)$ is labelled with a pair of indices where the index j indicates that $v_{jk} \in \mathcal{V}_j$. Thus,

$$\mathcal{V}_j = \{v_{jk} \mid k = 0, 1, \cdots, d_j\}. \tag{5}$$

By (4), the total number of vertices is

$$V = |\, \mathcal{V} \,| = (d_1 + 1) + \cdots + (d_n + 1) = d + n. \tag{6}$$

A more anatomical statement of FC is that "no face of P contains a bone". Our choice of the name "endoskeletal" for the class $\mathcal{E}^d$ reflects the fundamental structural role played by these internal bones.

Let F be a facet of a polytope $P \in \mathcal{E}^d$. By FC, $\mathcal{V}(F)$ must be missing at least one vertex from each bone. Since $|\mathcal{V}(F)| = d = V - n$, there is in fact exactly one vertex missing from $\mathcal{V}(F)$ from each bone. Thus, we can index the facets of P by the set of n-tples

$$\mathcal{K} = \{(k_1, ..., k_n) \mid 0 \leq k_j \leq d_j\}, \tag{7}$$

where for $\boldsymbol{k} \in \mathcal{K}$ the facet $F_{\boldsymbol{k}}$ is given by the (d–1)-simplex $F_{\boldsymbol{k}} = S(\mathcal{V}_{\boldsymbol{k}})$, where $\mathcal{V}_{\boldsymbol{k}} = \mathcal{V} \setminus \{v_{1k_1}, ..., v_{nk_n}\}$.

Our main result on the volume of an endoskeletal polytope is given in Theorem 3 in §5. While it might seem unfashionable to search for simple formulas for the volume of a polytope, we note that there

has been a recent surge of interest in the volume of polytopes among physicists [1].

2. Combinatorics

Given a polytope $P \in \mathcal{E}_\pi^d$, we can probe its endoskeleton by examining its f-vector $\boldsymbol{f}(P)$. A useful tool in this examination is the face number generating function

$$g_P(x) = \sum_{i=0}^{d} f_{i-1}(P) x^i, \tag{8}$$

where $f_{-1}(P) = 1$. According to FC (see (4)), each bone B_j can contribute $0, 1, \cdots, d_j$ of its $d_j + 1$ vertices to a face. Therefore,

$$g_P(x) = \prod_{j=1}^{n} \left[(1+x)^{d_j+1} - x^{d_j+1}\right]. \tag{9}$$

If we set $x = -1$ in (9) we obtain

$$\sum_{i=0}^{d} f_{i-1}(P)(-1)^i = \prod_{j=1}^{n} (-1)^{d_j} = (-1)^d,$$

or, moving the f_{-1} term to the right side, this more famous version:

$$\sum_{i=0}^{d-1} (-1)^i f_i(P) = 1 + (-1)^{d-1}, \tag{10}$$

namely, Euler's Equation [3].

Note also that

$$g_P(-x-1) = \prod_{j=1}^{n} (-1)^{d_j} \left[(1+x)^{d_j+1} - x^{d_j+1}\right],$$

whence,

$$g_P(-x-1) = (-1)^d g_P(x). \tag{11}$$

This identity encodes the Dehn-Sommerville generalization [3] of Euler's Equation. To see this, just plug (8) into (11):

$$\sum_{i=0}^{d} f_{i-1}(-x-1)^i = (-1)^d \sum_{k=0}^{d} f_{k-1} x^k. \tag{12}$$

The left side of (12) equals

$$\sum_{i=0}^{d} f_{i-1}(-1)^i \sum_{k=0}^{i} \binom{i}{k} x^k = \sum_{k=0}^{d} \sum_{i=k}^{d} (-1)^i \binom{i}{k} f_{i-1} x^k. \tag{13}$$

Comparing the right sides of (12) and (13), we obtain the Dehn-Sommerville equations: for $k = 0, \cdots, d$,

$$\sum_{i=k}^{d} (-1)^i \binom{i}{k} f_{i-1} = (-1)^d f_{k-1}, \tag{14}$$

of which the $k = 0$ equation is Euler's Equation.

This is not the only useful information which can be gleaned from the generating function (9). For instance, let $n_j(P)$ be the number of j-bones in the endoskeleton of $P \in \mathcal{E}_\pi^d$:

Lemma 1. *For any $P \in \mathcal{E}_\pi^d$,*

$$f_1(P) = \binom{V}{2} - n_1(P). \tag{15}$$

Proof: Given a polynomial $p(x) = p_0 + p_1 x + \cdots + p_m x^m$, let C_i be the map $C_i : p(x) \to p_i$. Now

$$f_1 = C_2\, g_P(x) = C_2 \{[(1+x)^2 - x^2]^{n_1} \prod_{j=1}^{n-n_1} (1+x)^{d_j+1}\},$$

where for $d_j \geq 2$ we have dropped the subtracted terms x^{d_j+1} in g_P because they don't affect the coefficient of x^2. We can also remove the subtracted terms x^2 provided we account explicitly for their net contribution:

$$f_1 = C_2 \prod_{j=1}^{n} (1+x)^{d_j+1} - n_1 = C_2\,(1+x)^V - n_1 = \binom{V}{2} - n_1.$$

$\square$

(15) has an obvious interpretation: the term $\binom{V}{2}$ gives the total number of possible edges of P and the term n_1 subtracts those which are missing by virtue of being a buried 1-bone. What (15) tells us is that there are no other "missing links", i.e., line segments joining a pair of vertices of P which are missing from $\mathcal{F}_1(P)$. Thus, despite our limited external viewpoint, we can still identify the 1-bones: The vertices subtending a 1-bone are precisely those which subtend $V-2$ edges rather than $V-1$ edges, and, if there are more than 2 such vertices, it is evident how they pair up in bones. We just look for the pairs of vertices not joined by a visible edge. We illustrate this with the hexahedron H^3 of Figure 2. H^3 has 2 bones: the 2-bone or triangle $B_1 = S(v_{10}, v_{11}, v_{12})$, and the 1-bone or line segment $B_2 = S(v_{20}, v_{21})$. Each vertex of the 1-bone B_2 subtends $V-2 = 3$ edges, whereas all the other vertices subtend $V-1 = 4$ edges.

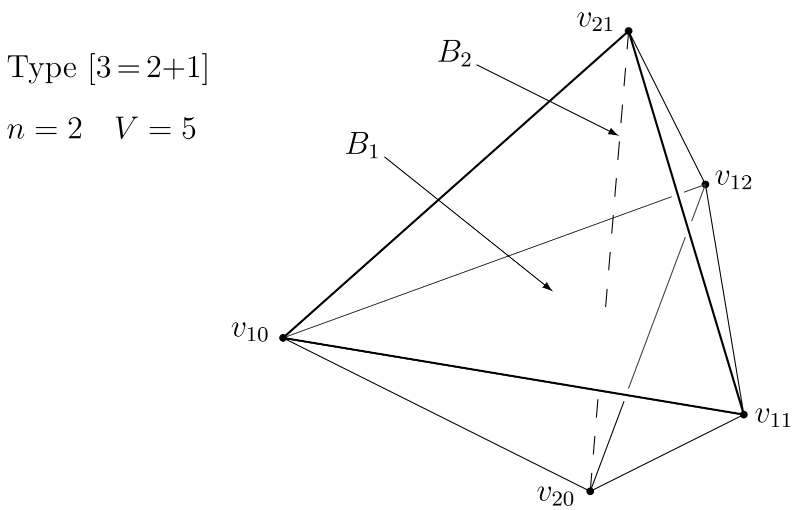


FIGURE 2. Hexahedron H^3

If we wish, we can push the reasoning of Lemma 1 further up the skeleton. For example, suppose that $f_1(P) = \binom{V}{2}$ so that P has no 1-bones. Then, as in Lemma 1, we can show that

$$f_2(P) = \binom{V}{3} - n_2(P). \tag{16}$$

If $n_1(P) > 0$ then the formula for $f_2(P)$ becomes more complicated.

We collect together some simple formulas for the endoskeletal face numbers:

Theorem 1. *Let $P \in \mathcal{E}^d_\pi$ where $\pi : d = d_1 + \cdots + d_n$.*
a) $f_0(P) = d + n$;
b) $f_1(P) = \binom{V}{2} - n_1(P)$;
c) $f_{d-1}(P) = (d_1 + 1)(d_2 + 1) \cdots (d_n + 1)$;
d) $f_{d-2}(P) = \frac{d}{2} f_{d-1}(P)$.

Proof: a) is (6); b) is (15).
c) Let $m_j(F)$ be the number of vertices chosen from $\mathcal{V}_j$ to form a facet F. By the Facet Property every choice with $m_j \le d_j$ and $\sum m_j = d$ is possible. No m_j can be strictly less than d_j or else $\sum m_j < \sum d_j = d$. Hence $m_j = d_j$ and the facets are formed by all possible choices of d vertices with one vertex left out from each $\mathcal{V}_j$. This gives the count c).
d) This identity holds in general for simplicial polytopes. Each facet is a $(d-1)$-simplex whose boundary consists of d $(d-2)$-simplices. Thus, $f_{d-2} = d\, f_{d-1}/2$ where we divide by 2 because each $(d-2)$-face is shared as a boundary simplex by 2 facets. $\square$

Table 1. Low Dimensional Endoskeletal Polytopes

d	π	n	f_0	f_1	f_2	f_3	f_4	Figure
2	2	1	3	1	-	-	-	triangle
2	1+1	2	4	4	-	-	-	quadrilateral
3	3	1	4	6	4	-	-	tetrahedron
3	2+1	2	5	9	6	-	-	hexahedron
3	1+1+1	3	6	12	8	-	-	octahedron
4	4	1	5	10	10	5	-	4-simplex
4	3+1	2	6	14	16	8	-	-
4	2+2	2	6	15	18	9	-	-
4	2+1+1	3	7	19	24	12	-	-
4	1+1+1+1	4	8	24	32	16	-	CP4
5	5	1	6	15	20	15	6	5-simplex
5	4+1	2	7	20	30	25	10	-
5	3+2	2	7	21	34	30	12	-
5	3+1+1	3	8	26	44	40	16	-
5	2+2+1	3	8	27	48	45	18	-
5	2+1+1+1	4	9	33	62	60	24	-
5	1+1+1+1+1	5	10	40	80	80	32	CP5

Table 1 compiles the various types of endoskeletal polytopes in dimensions 2 to 5 together with their face numbers. They illustrate the identities we have just established. Consider, for instance, the two tabulated types which have no 1-bones, namely $[4=2+2]$ and $[5=3+2]$. The first has $f_2 = 18$, $\binom{V}{3} = 20$ and $n_2 = 2$ and thus satisfies the identity (16); the second has $f_2 = 34$, $\binom{V}{3} = 35$ and $n_2 = 1$ and likewise satisfies (16).

Note also that no two different types in the table have the same f-vector $\boldsymbol{f}$. This uniqueness holds in general. For $\boldsymbol{f}$ determines the generating function whose set of zeros, given by

$$\{(e^{2\pi ik/(d_j+1)} - 1)^{-1} \mid j = 1, \cdots, n;\ k = 0, \cdots, d_j\},$$

determines n and $d_1, \cdots, d_n$.

If a convex polytope P has an endoskeletal f-vector, i.e., $\boldsymbol{f}(P)$ arises from some partition π of d, does it follow that the polytope is endoskeletal? Unfortunately, the answer is no. In order for a polytope P to be

endoskeletal, its vertices must satisfy a collective location constraint as well as the structural ones embodied by $\boldsymbol{f}(P)$. To understand this complication, consider the following example:

Example 1. *Shifted Octahedron*

Figure 3 displays the octahedron $\hat{O}^3$ which we have obtained from the endoskeletal octahedron O^3 of Figure 1 by sliding the axis $A = S(v_{30}\, v_{31})$ forward so that it passes in front of the edge $S(v_{10}\, v_{20})$ rather than passing through one of the 2-simplices $S(v_{10}\, v_{11}\, v_{20})$ or $S(v_{10}\, v_{11}\, v_{21})$. As a result, A is no longer buried within the octahedron $\hat{O}^3$. $\hat{O}^3$ still has 3 buried edges like O^3, and the same f-vector, namely, $\boldsymbol{f} = (6, 12, 8)$.

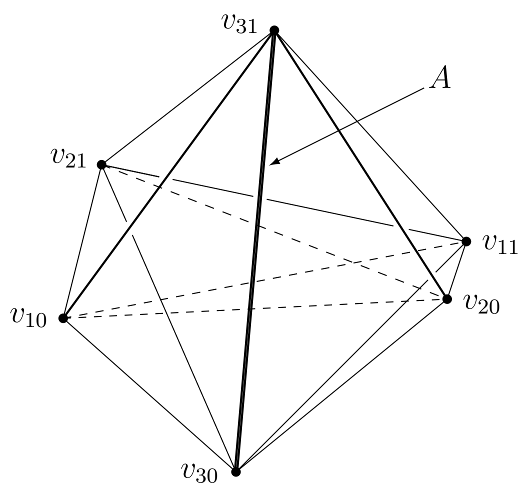


FIGURE 3. Shifted Octahedron $\hat{O}^3$

Unfortunately, $\hat{O}^3$ is no longer endoskeletal, as can be seen either from the fact that v_{10} and v_{20} subtend only $3 = V - 3$ edges or from the fact that the 3 pairs of endpoints of the buried edges of $\hat{O}^3$ (the dashed line segments) do not provide a partition of $\mathcal{V}$. Moreover, $\hat{O}^3$ is not determinantal (i.e., its volume is not given by a single determinant) as we explain at the end of Section 7.

Remark 1. *Identifying the endoskeletals*

To tell whether a given convex polytope P is an endoskeletal, we need to count:
1. If any vertex subtends fewer than $V - 2$ edges, then P is not an endoskeletal.

2. If $\boldsymbol{f}(P)$ is not in the list of known endoskeletal f-vectors or if it fails to satisfy the known identitites, such as those in Theorem 2, then P is not endoskeletal. If $\boldsymbol{f}(P)$ passes these tests, then we go to 3).
3. By examining those vertices which subtend $V-2$ edges, we find all the pairs, $p_1, \cdots, p_{n_1}$ of vertices which are not joined by an edge of P. By Test 1, $p_1, \cdots, p_{n_1}$ are mutually disjoint, and $S(p_1), \cdots, S(p_{n_1})$ are the potential 1-bones. Letting $\mathcal{V}_1 = \mathcal{V} \setminus \{p_1, \cdots, p_{n_1}\}$, we go to 4).
4. Find all the triples in $\mathcal{V}_1$ which are not the vertices of a 2-face. If these triples are not mutually disjoint, then P is not endoskeletal. If they are, etc.
5. We continue in this way, prospecting for disjoint bones of increasing dimension. If the mining operation exhausts all the vertices in $\mathcal{V}$, then P is an endoskeletal.

The octahedron $\hat{O}^3$ of Example 1 passes Test 2 but fails Test 1.

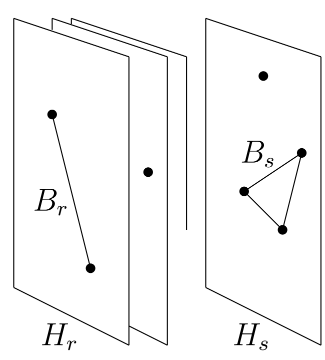
3. Standard Form

Given $P \in \mathcal{E}^d_\pi$, we define its axes to be the differences of its vertex vectors, $a_{jk} = v_{jk} - v_{j0},\ k = 1, \cdots, d_j$. The set of all axes,

$$\mathcal{A}(P) = \bigcup_{j=1}^{n} \mathcal{A}_j(P) = \bigcup_{j=1}^{n} \{a_{jk} \mid k = 1, \cdots, d_j\} \tag{17}$$

satisfies:

Lemma 2. *If $P \in \mathcal{E}^d_\pi$ then the d vectors in $\mathcal{A}(P)$ are linearly independent.*



Proof: Letting $H = \operatorname{span} \mathcal{A}$, suppose on the contrary that $\dim H < d$. We examine the case $\dim H = d-1$ (the lower dimensional cases are similar). Consider the stack of hyperplanes, $H_j = H + v_{j0}$, where $j = 1, \cdots, n$ (they are not necessarily distinct). Clearly, the bone $B_j \subset H_j$. Let H_r and H_s be the two planes at the ends of the stack. They are both supporting planes for P and hence the bones B_r and B_s lie on a facet of P, contradicting FC. $\square$

Lemma 2 has several useful consequences:
1) We can take $\mathcal{A}$ as a basis for $\mathbb{R}^d$.

2) The fact that the vectors $a_{j1}, \cdots, a_{jd_j}$ are linearly independent, or, equivalently, that the vertices in $\mathcal{V}_j$ are affinely independent, means that the simplex $B_j = S(\mathcal{V}_j)$ has dimension d_j.
3) We can reduce a general endoskeletal polytope P to a simpler standard form as follows. Let A be the $d \times d$ matrix whose columns are the vectors of $\mathcal{A}$ in lexicographic order. If we change coordinates by $x \to x' = A^{-1}x$, we obtain

$$a_{jk} \longrightarrow a'_{jk} = e_m \tag{18}$$

where e_m is the standard basis vector with

$$m = m(j,k) = d_1 + \cdots + d_{j-1} + k. \tag{19}$$

Thus the transformed polytope P' has the standard basis vectors as its vertex differences. We say that P' is in standard form and we write $\mathcal{T}_\pi^d$ for the class of such polytopes. Since a linear transformation does not change the nature of the various objects under consideration (e.g., a j-simplex remains a j-simplex and a hyperplane remains a hyperplane) nor the relations among them, such as intersection and inclusion, we can assume that our polytopes are in standard form.

The best way to try to comprehend the incomprehensible world of $d > 3$ dimensions is to make a strategic use of projections into lower dimensions. In particular, we shall deploy projections onto a 2-plane and along a bone. Although we could project along the general basis vectors a_{jk}, for visualization purposes we find it best to put the polytope into standard form and to make orthogonal projections along the standard basis vectors. When we project onto a $\tilde{d}$-plane Π where $\tilde{d} < d$ we shall identify Π with $\mathbb{R}^{\tilde{d}}$ (in other words, we shall ignore the 0 components).

If B_i and B_j are two bones of $P \in \mathcal{T}_\pi^d$, let $r = m(i,1)$ and $s = m(j,1)$ so that $a_{i1} = e_r$ and $a_{j1} = e_s$. Suppose P is projected onto $\tilde{P}$ in the $x_r x_s$ plane. Then the image $\tilde{B}_i$ of B_i is a unit line segment parallel to the x_r-axis, the image $\tilde{B}_j$ is a unit line segment parallel to the x_s-axis, and all the other bones go into points. Now we invoke FC. We say that a bone is "exposed" in P if it lies on a facet and otherwise we say it is "buried", and we use the same words for the projected objects. Obviously, if a bone B is buried in P then its projection $\tilde{B}$ must be buried in $\tilde{P}$ (the converse is false). By FC all the bones are buried and since a point bone can't cover other bones without exposing itself, it follows that $\tilde{B}_i$ and $\tilde{B}_j$ must cross one another and that the other bones must be mapped into points interior to the quadrilateral $\tilde{Q}$ with axes $\tilde{B}_i$ and $\tilde{B}_j$. We suppress the line segments ending in a point bone

(the images of the other faces of P).Thus $\tilde{P}$ has the appearance of a "spotted kite":

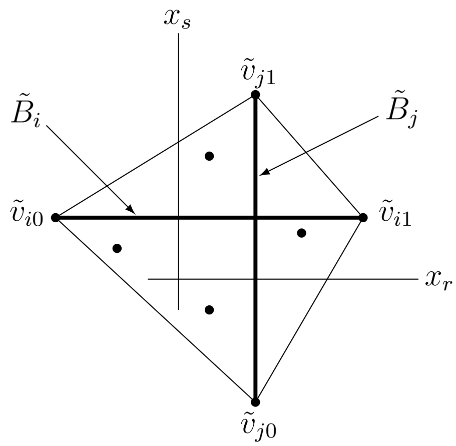


FIGURE 4. $\tilde{Q}$ in the $x_r x_s$-plane

We next describe the projection of $P \in \mathcal{T}_\pi^d$ along a bone, say, B_n. Let $\tilde{d} = d - d_n$. We define the orthogonal projection R in $\mathbb{R}^d$ by

$$R\, e_m = \begin{cases} e_m & \text{if} \quad m \leq \tilde{d} \\ 0 & \text{if} \quad m > \tilde{d} \end{cases} .$$

Let $\tilde{v}_{jk} = R\, v_{jk}$, $\tilde{\mathcal{V}}_j = R\, \mathcal{V}_j = \{\tilde{v}_{jk} \mid k = 0, \cdots, d_j\}$, $\tilde{B}_j = S(\tilde{\mathcal{V}}_j)$, $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_1 \cup \cdots \cup \tilde{\mathcal{V}}_{n-1}$ and $\tilde{P} = \operatorname{conv} \tilde{\mathcal{V}}$. The polytope $\tilde{P}$ is contained in the $x_1 \cdots x_{\tilde{d}}$-plane which we identify with $\mathbb{R}^{\tilde{d}}$. The vertices $\mathcal{V}_n$ of B_n are all mapped into the same point $\tilde{v}_{n0}$ which must be in $\tilde{P}^\circ$ since B_n is buried in P.

Lemma 3. *If $P \in \mathcal{T}_\pi^d$, then $\tilde{P} \in \mathcal{T}_{\tilde{\pi}}^{\tilde{d}}$ where $\tilde{\pi} : \tilde{d} = d_1 + \cdots + d_{n-1}$ and the corresponding partition of $\tilde{\mathcal{V}} = \mathcal{V}(\tilde{P})$ is $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_1 \cup \cdots \cup \tilde{\mathcal{V}}_{n-1}$.*

Proof: Note first that the vertices in $\tilde{\mathcal{V}}$ are convexly independent, where by this we mean that none of them is a convex combination of the others. Suppose, on the contrary, that $\tilde{v}_{i1}$, say, is a convex combination of the other vertices in $\tilde{\mathcal{V}}$. Let e_s be the axis vector in the direction of $\tilde{a}_{i1} = \tilde{v}_{i1} - \tilde{v}_{i0}$. We regard this direction as the "up" direction. All the other direction vectors are orthogonal to the up direction and at least one of them, say, $\tilde{a}_{jk}$, must be at least as "high" as $\tilde{v}_{i1}$ or else $\tilde{v}_{i1}$ couldn't be a convex combination of other vertices. Let e_r be the axis vector in the direction of $\tilde{a}_{jk}$. Now project the polytope onto the $x_r x_s$-plane. The two unit line segments corresponding to the two direction vectors $\tilde{a}_{jk}$ and $\tilde{a}_{i1}$ form a $\top$ shape (where there may be a

gap between the two line segments) rather than the required cross of Figure 4. Similarly, if we assume that it is a "base" vertex $\tilde{v}_{i0}$ which is a convex combination of other vertices we arrive at a contradiction with a shape like $\perp$.

It is easy to see that FC survives the projection process. For suppose, on the contrary, that a vertex set $\tilde{\mathcal{V}}_j$, $j < n$, is contained in a facet $\tilde{F} \in \mathcal{F}_{\tilde{d}-1}(\tilde{P})$. Let Z be the null space of Q. Now $\tilde{F} \subset \tilde{H}$ where $\tilde{H}$ is a $(\tilde{d}-1)$-supporting plane for $\tilde{P}$. Let H be the pre-image of $\tilde{H}$, $H = \{\tilde{x}+z \mid \tilde{x} \in \tilde{H},\ z \in Z\} \subset \mathbb{R}^d$. Then H is a supporting hyperplane for P containing $\mathcal{V}_j$. This violates FC for P. Conversely, suppose that $\tilde{U}$ is a set of $\tilde{d}$ vertices of $\tilde{P}$ which contains no $\tilde{\mathcal{V}}_j$, $j < n$. By the usual counting, $\tilde{U}$ must be missing exactly one vertex from each $\tilde{\mathcal{V}}_j$, $j < n$, i.e., $\tilde{U}$ has the form

$$\tilde{U} = \tilde{\mathcal{V}}_{\boldsymbol{l}} = \tilde{\mathcal{V}} \setminus \{\tilde{v}_{1l_1}, ..., \tilde{v}_{n-1,l_{n-1}}\} \tag{20}$$

where

$$\boldsymbol{l} \in \mathcal{L} = \{(l_1, ..., l_{n-1}) \mid 0 \le l_j \le d_j, j = 1, \cdots, n-1\}. \tag{21}$$

Thus, $S(\tilde{U})$ has precisely the form of a facet of $\tilde{P}$. □

4. Partition Schemes

We make use of two different schemes for partitioning a polytope $P \in \mathcal{E}_\pi^d$ into d-simplices:

a) Backbone scheme We single out a particular bone, say B_n, as the "backbone" and require that the vertex set of each simplex in the partition consists of all the vertices of the backbone and all the vertices but one of each of the other bones. By (4) this gives a total of $d + n - (n-1) = d+1$ vertices, as is required for a d-simplex. Thus the backbone partition is given by

$$P = \bigcup_{\boldsymbol{l} \in \mathcal{L}} S_{\boldsymbol{l}}, \tag{22}$$

where the simplex $S_{\boldsymbol{l}} = S(\mathcal{V}(P) \setminus \{v_{1l_1}, ..., v_{n-1,l_{n-1}}\})$ and $\mathcal{L}$ is defined in (21).

b) Focal scheme We place an auxiliary point f inside P (the "focus") and for each facet $F_{\boldsymbol{k}}$ of P, as defined in (7), we form the d-simplex $S_{\boldsymbol{k}}^f$ with base $F_{\boldsymbol{k}}$ and apex vertex f. The focal partition is given by

$$P = \bigcup_{\boldsymbol{k} \in K} S_{\boldsymbol{k}}^f. \tag{23}$$

We illustrate the two schemes for a quadrilateral Q (type [2=1+1]):

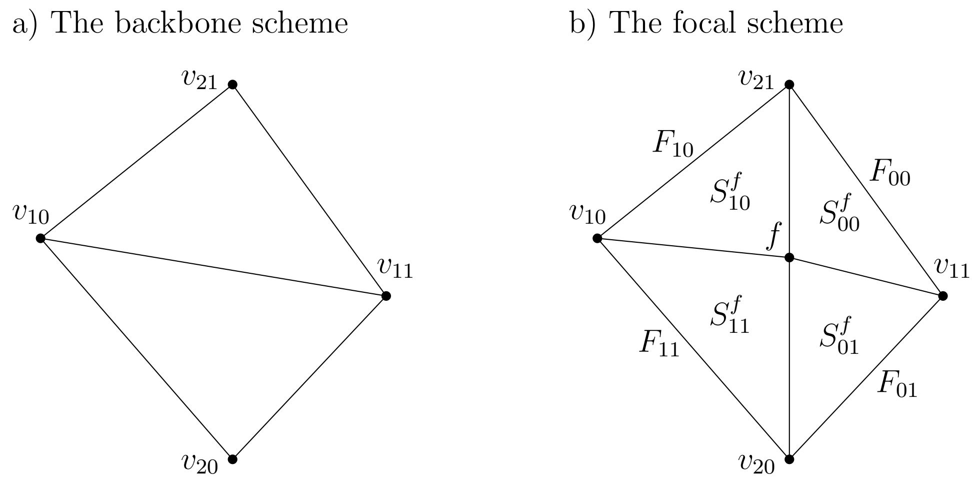


FIGURE 5. Partition schemes for Q

The first scheme has 2 partition simplices (triangles) and the second scheme has 4. While the backbone scheme may look more efficient, as indeed it would be in a numerical calculation, theoretically it has a disadvantage, as we now explain. In any scheme, the partition simplices must cover P and must be "essentially disjoint", where by this we mean that their interiors are disjoint (they can of course intersect on their boundaries). These properties are completely obvious for the focal scheme, whereas in the backbone scheme they require nontrivial proofs (given in the following lemma). In particular, the proof of disjointness must be piggybacked on the known disjointness of the focal scheme.

Lemma 4. *Consider the backbone partition scheme (22) for $P \in \mathcal{E}^d$.*
a) $P = \bigcup_{\boldsymbol{l} \in \mathcal{L}} S_{\boldsymbol{l}}$.
b) The simplices $S_{\boldsymbol{l}}$, $\boldsymbol{l} \in \mathcal{L}$, are (essentially) disjoint.

Proof: a) We wish to show that every point $x \in P$ belongs to some simplex $S_{\boldsymbol{l}}$. First consider a face F of P. For each $j = 1, ..., n-1$, $B_j \not\subset F$ and so there must be a vertex v_{jl_j} of B_j which is not in F. It follows that $F \subset S_{\boldsymbol{l}}$ where $\boldsymbol{l} = (l_1, ...l_{-1}n)$. Next let $x \in P^\circ$, the interior of P. Take any $v \in \mathcal{V}_1 \subset \partial P$. Now the line L passing through x and v must cross ∂P in a second point y which, as we have just noted, must be in some $S_{\boldsymbol{l}}$. Since $\mathcal{V}_1 \subset S_{\boldsymbol{l}}$ for all $\boldsymbol{l}$, v is also in $S_{\boldsymbol{l}}$ and hence, as a convex combination of v and y, so is x.

b) Suppose that P is in standard form. We project along the backbone B_n to collapse it to a point f (see Lemma 3). Then P with its backbone partition scheme projects down to a lower dimensional polytope $\tilde{P}$ with

a focal partition scheme with focus f. Each d-simplex S_ι projects down to a $\tilde{d}$-simplex $\tilde{S}_\iota$ where $\tilde{S}_\iota = S(\tilde{\mathcal{V}}_\iota \cup \{f\})$ with $\tilde{\mathcal{V}}_\iota$ defined in (20) and $\tilde{d} = d - d_n$. The disjointness of the S_ι's follows from that of the $\tilde{S}_\iota$'s. $\square$

5. The Volume formula

To derive our formula for the volume of an endoskeletal polytope, we shall make use of the following basic fact about conical figures. Consider the d-cone C formed by joining an apex vertex v to the points of a base object B lying in a $(d-1)$-dimensional plane H. Then

$$\operatorname{vol}_d C = \frac{h}{d} \operatorname{vol}_{d-1} B, \tag{24}$$

where h is the distance from v to H. (24) follows from a scaling argument: If we slice C with planes $H(t)$ parallel to H and a distance t from H, where $0 \leq t \leq h$, then the cross-section $B(t)$ is similar to B but is scaled by a factor $(h-t)/h$. Hence, $\operatorname{vol}_{d-1} B(t) = (\frac{h-t}{h})^{d-1} \operatorname{vol}_{d-1} B$, and integration with respect to t yields (24).

An immediate consequence of (24) is that if the apex vertex v of a cone C is translated parallel to a vector in the base plane then $\operatorname{vol} C$ is constant. For a simplex this constancy is also obvious from the basic volume formula (2). It is convenient to rewrite (2) in a more symmetric form by using the augmented vectors $\hat{v} = \binom{v}{1} \in \mathbb{R}^{d+1}$:

$$\operatorname{vol} S = \frac{1}{d!} |\det(\hat{v}_0, \hat{v}_1, \cdots, \hat{v}_d)|. \tag{25}$$

Note that one way of defining the affine independence of $v_1, \cdots, v_m$ is to say that $\hat{v}_0, \hat{v}_1, \cdots, \hat{v}_m$ are linearly independent.

Some further notation: If $\mathcal{F} = \{v_1, \cdots, v_n\}$ is a set of n vectors in $\mathbb{R}^d$, we denote the $d \times n$ matrix $(v_1, \cdots, v_n)$ by $M(\mathcal{F})$ and the $(d+1) \times n$ matrix $(\hat{v}_1, \cdots, \hat{v}_n)$ by $\hat{M}(\mathcal{F})$.

Consider a $P \in \mathcal{T}^d_\pi$ with bones $B_j = (v_{j0} \cdots v_{jd_j}),\ j = 1, \cdots, n$, and

$$v_{nd_n} - v_{n0} = e_d. \tag{26}$$

We wish to compute $\operatorname{vol} P$ inductively by peeling off e_d, but what prevents us from doing so directly is that the vertices $\mathcal{V}_{<n}$ of $B_1, \cdots, B_{n-1}$ do not necessarily lie in a hyperplane with normal e_d. The hyperplane we would like to use is the one with normal e_d passing through v_{n0}:

$$H = \{x \mid f_d(x) \equiv (x - v_{n0}) \cdot e_d = 0\}.$$

Whereas the vertices $v_{n0} \cdots v_{nd_{n-1}}$ do lie on H, we know from our discussion around Figure 5 that the vertices $v \in \mathcal{V}_{<n}$ may be scattered

"above" H in the strip $\{x \mid 0 \leq f_d(x) < 1\}$. To finesse this problem we hold v_{nd_n} fixed and lower the vertices $\mathcal{V}_{<n}$ onto H via the mapping

$$x \longrightarrow x'(t) = x - t(x - v_{n0})\cdot e_d\, e_d, \tag{27}$$

where $0 \leq t \leq 1$. We choose a backbone partition with backbone the fixed bone B_n.

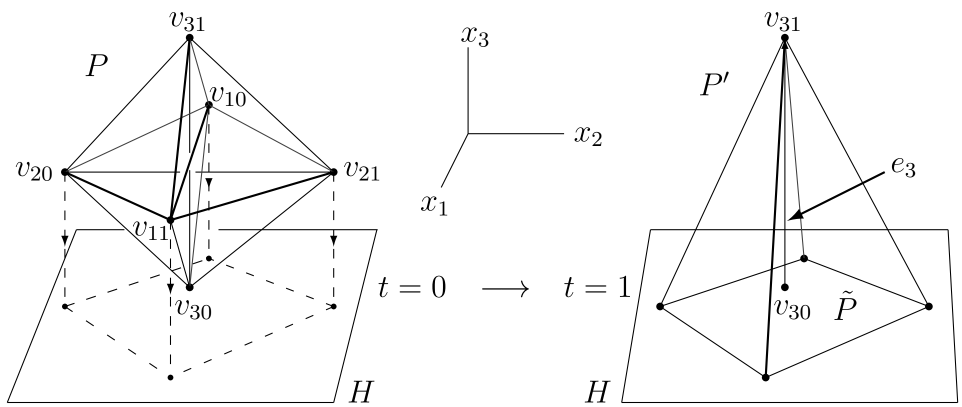


FIGURE 6. Lowering the Octahedron [1+1+1]

Let P' be the image of P under (27). For $t < 1$ the mapping $P \to P'$ is 1-1 so that P' is in $\mathcal{T}_\pi^d$ and inherits the partition of P. When $t=1$ all the vertices of P' lie on H except for v_{nd_n} so that P' is conical, but it is no longer endoskeletal inasmuch as all its bones are exposed. It is easy to verify that the vertices on H form a $(d-1)$-polytope $\tilde{P}$ in $\mathcal{T}_{\tilde{\pi}}^{d-1}$, where $\tilde{\pi}$ is the partition of $d-1$ obtained from π by replacing its last summand d_n by d_n-1. The key feature of this procedure is that

$$\operatorname{vol} P' = \operatorname{vol} P, \tag{28}$$

for the simple reason that each simplex in the partition of P' has e_d as a fixed axis. Consequently, its volume is invariant under a translation of any of the other vertices in the x_d direction. We immediately obtain:

Theorem 2. *For* $P \in \mathcal{T}_\pi^d$,

$$vol\, P = \frac{1}{d!}. \tag{29}$$

Proof: (29) follows by induction from (24) and (28). □

Theorem 3. (**Volume Formula I**) *If* $P \in \mathcal{E}^d_\pi$, *then*

$$vol\, P = \frac{1}{d!} \mid det\, M(\mathcal{A}) \mid , \tag{30}$$

where $\mathcal{A}$ *is the set of axis vectors of* P *given in (17).*

Proof: The reduction (18) of $P \in \mathcal{E}^d_\pi$ to $P' \in \mathcal{T}^d_\pi$ produces the volume factor $|\det A| = |\det M(\mathcal{A})|$. Thus (30) follows at once from (29). $\square$

The volume formula (30) can be recast directly in terms of the vertices of P in the following way. Consider the $d \times V$ matrix

$$M = M(\mathcal{V}(P)) = (M(\mathcal{V}_1), \cdots, M(\mathcal{V}_n)). \tag{31}$$

We augment M with an additional $n = V - d$ rows to form a $V \times V$ matrix

$$M_E = \begin{pmatrix} M(\mathcal{V}_1) & M(\mathcal{V}_2) & \cdots & M(\mathcal{V}_n) \\ E_1 & E_2 & \cdots & E_n \end{pmatrix}, \tag{32}$$

where $E_j = (e_j, \cdots, e_j)$ is the $n \times (d_j + 1)$ matrix each of whose $d_j + 1$ columns is the jth standard basis vector for $\mathbb{R}^n$. For example, suppose P is of Type [5=2+2+1]. Then

$$M_E = \begin{pmatrix} v_{10} & v_{11} & v_{12} & v_{20} & v_{21} & v_{22} & v_{30} & v_{31} \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

If we compute $|\det M_E|$ using the column operations $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_5 \to C_5 - C_4$, etc., we obtain the 5×5 determinant:

$$|\det(a_{11}\ a_{12}\ a_{21}\ a_{22}\ a_{31})| = |\det M(\mathcal{A})|,$$

as in (30). The same reasoning works in general to give this version of (30):

Corollary 1. (**Volume Formula II**) *If the polytope* $P \in \mathcal{E}^d_\pi$, *then*

$$vol\, P = \frac{1}{d!} \left| \det M(\mathcal{V}(P))_E \right| . \tag{33}$$

Remark 2. *Translation invariance*

One remarkable consequence of (30) is that $\mathrm{vol}\, P$ is invariant under a translation of each bone. At first sight this seems preposterous! After all, translating one bone far away from the other bones ought to increase $\mathrm{vol}\, P$ arbitrarily. However, we must not forget that the positions of the bones are severely restricted by the Facet Condition (FC) of (4). Moving a bone far enough will eventually expose it and violate

FC (see Figure 7 below). What we can say is that $\operatorname{vol} P$ is invariant under small translations of the bones. We shall give a more detailed analysis in Section 7.

Remark 3. *Connection with the Backbone Partition*

To understand how the backbone scheme partition (22) arises, consider the volume formula (30). We single out the bone B_1 and take $v_{10} = 0$ (any other bone and vertex could be chosen and there is no loss of generality in setting $v_{10} = 0$). Then in (30)

$$\det M(\mathcal{A}) = \det(v_{11}, \cdot\cdot, v_{1d_1}, v_{21} - v_{20}, \cdot\cdot, v_{2d_1} - v_{20}, \cdot\cdot, v_{n1} - v_{n0}, \cdot\cdot\, v_{nd_n} - v_{n0}).$$

We next expand out the differences using the linearity of the determinant. The terms with a repeated base vertex v_{j0} vanish. The nonzero terms are obtained by choosing d_j different vertices from the bone B_j for each $j = 2, \cdots, n$. We are thus left with

$$\det M(\mathcal{A}) = \sum_{\boldsymbol{l} \in \mathcal{L}} \sigma_{\boldsymbol{l}} \det M(\mathcal{V}_{\boldsymbol{l}}), \tag{34}$$

where

$$\mathcal{L} = \{(l_2, ..., l_n) \mid 0 \leq l_j \leq d_j,\ j = 2, \cdots, n\}, \tag{35}$$

$\mathcal{V}_{\boldsymbol{l}} = \mathcal{V} \setminus \{v_{2l_2}, ..., v_{nl_n}\}$, and the signs $\sigma_{\boldsymbol{l}} = \pm 1$ arise from the reordering of columns in the determinants.

This is the backbone partition scheme! By (2), $\frac{1}{d!}|\det M(\mathcal{V}_{\boldsymbol{l}})| = \operatorname{vol} S_{\boldsymbol{l}}$ where $S_{\boldsymbol{l}} = S(\mathcal{V}_{\boldsymbol{l}})$ is a simplex in the backbone partition scheme. Now we know from Lemma 4 that $\operatorname{vol} P = \sum_{\boldsymbol{l} \in \mathcal{L}} \operatorname{vol} S_{\boldsymbol{l}}$. Hence, the signs in (34) must be coherent in the sense that (34) reads

$$\sigma \det M(\mathcal{A}) = \sum_{\boldsymbol{l} \in \mathcal{L}} \sigma_{\boldsymbol{l}} \det M(\mathcal{V}_{\boldsymbol{l}}), \tag{36}$$

where $\sigma = \operatorname{sign} \det M(\mathcal{A})$ and $\sigma_{\boldsymbol{l}} = \operatorname{sign} \det M(\mathcal{V}_{\boldsymbol{l}})$. We can summarize the above reasoning in the equation

$$\operatorname{vol} P = \sum_{l \in \mathcal{L}} \operatorname{vol} S_{\boldsymbol{l}} = \frac{1}{d!} \sum_{l \in \mathcal{L}} |\det M(\mathcal{V}_{\boldsymbol{l}})| = \frac{1}{d!} |\det M(\mathcal{A})|. \tag{37}$$

6. Existence

The restrictions imposed by FC raise the existential question: Given an arbitrary partition π of d, does there actually exist a polytope of Type $[\pi]$? We settle this question here with an explicit construction of a polytope $P \in \mathcal{T}^d_\pi$.

Let $\{a_{jk}\},\ j=1,\cdots,n,\ k=1,\cdots,d_j$ be the standard basis vectors for $\mathbb{R}^d$, which were denoted a'_{jk} in (18). Let R_j be the subspace spanned by $\mathcal{A}_j=\{a_{jk}\mid k=1,\cdots,d_j\}$ so that

$$\mathbb{R}^d = R_1+\cdots+R_n. \tag{38}$$

We define the bone B_j to be the d_j-simplex in R_j,

$$B_j = \operatorname{conv}\mathcal{V}_j = \operatorname{conv}\{v_{jk}\mid\ k=0,\cdots,d_j\}, \tag{39}$$

where $v_{jk}=v_{j0}+a_{jk},\ k=1,\cdots,d_j$ and $v_{j0}=-\frac{1}{d_j+1}\sum_{k=1}^{d_j}a_{jk}$. It's easy to see that $\mathcal{V}=\bigcup\mathcal{V}_j$ is convexly independent and is thus a legitimate vertex set.

Lemma 5. *$P\equiv conv\,\mathcal{V}\in\mathcal{T}^d_\pi$.*

<u>Proof</u>: Note that $V\equiv|\mathcal{V}|=d+n$. With the choice (39), each bone B_j is centred at 0, as is P. Since it contains an interior point of P, B_j cannot lie on a facet F of P. We make the usual count: Each $\mathcal{V}(F)$ must be missing at least one vertex from each bone and therefore at least n vertices altogether. Since $|\mathcal{V}(F)|\geq d$, $\mathcal{V}(F)$ can be missing at most n vertices, and so $|\mathcal{V}(F)|=d$ with one vertex missing from each bone. Therefore, F is a $(d-1)$-simplex and P is simplicial.

Let $\mathcal{U}\subset\mathcal{V}$, $|\mathcal{U}|=d$. The FC states that $S(\mathcal{U})$ is a facet of P if and only if $\mathcal{U}$ is missing one vertex from each bone. To prove the converse of FC suppose that $\mathcal{U}$ is missing one vertex from each bone, i.e., for each $j=1,\cdots,n$, $\mathcal{V}_j\setminus\mathcal{U}=\{v_{jk_j}\}$. Let H be the hyperplane spanned by the (linearly independent) vectors in $\mathcal{U}$ and let Λ be the open half-space on the side of H containing 0. Consider any v_{jk_j}. Since all the other vertices in $\mathcal{V}_j$ lie on H and since $0\in B_j$ it must be that $v_{jk_j}\in\Lambda$. Therefore, $\{v_{1k_1},\cdots,v_{nk_n}\}\subset\Lambda$, whence H is a supporting plane for P and $S(\mathcal{U})\in\mathcal{F}_{d-1}$. □

Note that the P we have just constructed is "pinned" in the sense that all its bones pass through a common point (the origin). However, when $n>2$ there is enough "wiggle room" to unpin it. By this we mean that we can slightly displace the rib bones $B_j\to B'_j$, so that no two of the displaced bones $B'_1,\cdots,B'_n$ meet one another. Consider the following displacement of the vertices $v_{j0},\ j=2,\cdots,n$:

$$v_{j0}\ \to\ v'_{j0}=v_{j0}+a(b_{j-1}+b_{j+1}), \tag{40}$$

where $a\neq 0$ and $b_j=b_{j1}$ with $b_{n+1}=0$. Let $B'_j=S(\{v'_{j0}\}\cup\mathcal{V}_j/\{v_{j0}\})$ be the displacement of B_j. The effect of (40) is to give every interior

point of B_j' a nonzero component in R_{j-1} and R_{j+1}. In the matrix

$$\begin{pmatrix} - & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathsf{X} & - & \mathsf{X} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathsf{X} & - & \mathsf{X} & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & \mathsf{X} & - & \mathsf{X} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathsf{X} & - \end{pmatrix},$$

an X (respectively 0) entry in the jk-position indicates that every interior point of B_j' has a nonzero (respectively zero) component in the subspace R_k. For example, the second row shows that every interior point of B_2' has a nonzero component in the subspaces R_1 and R_3 and a zero component in $R_4, \cdots, R_n$. Clearly, for any two rows there is a column in which one of the rows has a 0 entry and the other an X entry. Thus, the bones associated with the two rows do not have a point in common, and we conclude that $B_1', \cdots, B_n'$ are mutually disjoint.

7. Further Examples

Thus far, all of our examples of determinantal polytopes have been endoskeletal. Are there any others? In this final section, we construct examples of determinantal polyhedra which are not simplicial or convex and therefore not endoskeletal. These constructions are nevertheless based on our endoskeletal results inasmuch as they consist of perturbing an endoskeletal polytope. Our discussion is entirely exemplary - we make no attempt to compile a complete taxonomy of determinantal polyhedra. (We use the term "polyhedron" to refer to an object which is not convex.)

Example 2. *Non-simplicial determinantal polytopes*

Let π be a non-trivial partition of $d-1$, and let $P \in \mathcal{E}_\pi^{d-1}$ be an endoskeletal polytope in $\mathbb{R}^{d-1}$. We identify $\mathbb{R}^{d-1}$ with the hyperplane $H = \{x \in \mathbb{R}^d \mid x_d = 0\}$ in $\mathbb{R}^d$. Let v be any point in $\mathbb{R}^d$ not on H. Then the cone C with base $P \subset H$ and apex vertex v is our desired non-simplicial determinantal polytope. Clearly, C is convex but non-simplicial because its facet P is not a simplex (π being non-trivial). As for its volume, by (24) and Theorem 3,

$$\operatorname{vol}_d C = \frac{h}{d} \operatorname{vol}_{d-1} P = \frac{h}{d!} |\det M|, \tag{41}$$

where $h = \operatorname{dist}(v, H)$ and the columns of the $(d-1) \times (d-1)$ matrix $M = M(\mathcal{A})$ are the axis vectors $\mathcal{A} = \{a_{jk}\} \subset \mathbb{R}^{d-1}$ of P. Let

$\hat{a}_{jk} = \binom{a_{jk}}{0} \in \mathbb{R}^d$. By Lemma 2 the vectors $\{a_{jk}\}$ form a basis for $\mathbb{R}^{d-1}$ so that for any vertex v_{lm} of P

$$v - v_{lm} = \sum c_{jk}\, \hat{a}_{jk} \,\pm\, he_d\,.$$

Hence,

$$(42)\quad |\det(\hat{a}_{10}\ \hat{a}_{11}\ \cdots\ v{-}v_{lm})| = |\det(\hat{a}_{10}\ \hat{a}_{11}\ \cdots\ \begin{pmatrix}0\\ h\end{pmatrix})| = h\,|\det M|.$$

We conclude from (41) and (42) that C is determinantal.

It is unlikely that there are determinantal polytopes with two or more non-simplicial facets. For suppose that there is a determinantal polytope P with two non-simplicial facets F_1 and F_2 whose vertex sets are located in distinct blocks in the formula for $\operatorname{vol} P$. If we shrink F_2, say, to a point v, then the volume formula gives 0 but the polytope still has positive volume as it contains the cone with base F_1 and apex vertex v.

For a polytope $P \in \mathcal{E}^d_\pi$, let $\mathcal{H}(P)$ be the set of hyperplanes in $\mathbb{R}^d$ which contain d or more vertices of P. Let $\mathcal{H}_S(P) \subset \mathcal{H}(P)$ be those which are supporting planes for P and $\mathcal{H}_I(P) = \mathcal{H}(P) \setminus \mathcal{H}_S(P)$ be those which are internal to P. For $H \in \mathcal{H}$ let $\mathcal{V}(H) = \mathcal{V}(P) \cap H$. If we move the vertices of P around, then, as long as a vertex v does not cross a hyperplane in $\mathcal{H}(P)$, P stays in $\mathcal{E}^d_\pi$ and the volume formula (30) remains valid. However, the outcome is different when a crossing occurs, as the following simple example shows:

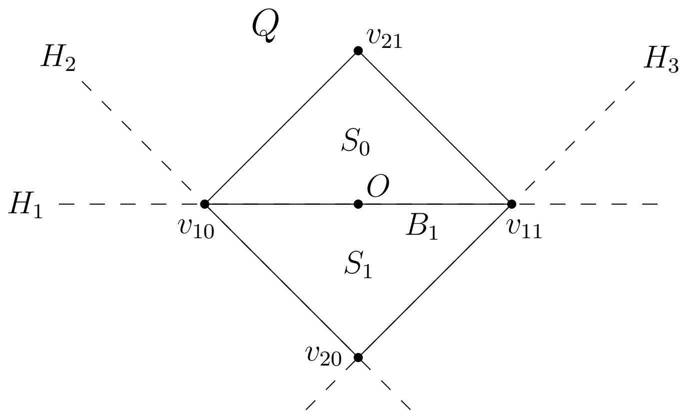
Example 3. *Quadrilateral Q*

We take Q of Type [2=1+1] in standard form. As we intend to relocate the vertex $v_{21} = (0, .5)$ in various ways, we display (with dashed lines) only those hyperplanes in $\mathcal{H}(Q)$ which do not contain v_{21} and thus remain fixed, namely,

$$H_1 = \{x \mid x_2 = 0\} \in \mathcal{H}_I,$$

and the supporting hyperplanes

$$H_2 = \{x \mid x_2 + x_1 + .5 = 0\}, \quad H_3 = \{x \mid x_1 - x_2 + .5 = 0\}\,:$$



As in Figure 5, we choose $B_1 = S(v_{10}, v_{11})$ as the backbone and partition the quadrilateral as $Q = S_0 \cup S_1$ where $S_0 = S(v_{10}, v_{11}, v_{21})$ and $S_1 = S(v_{10}, v_{11}, v_{20})$. To allow the modified quadrilateral Q' to be non-convex, when we relocate v_{21} to v'_{21} we do not form Q' as the convex hull of its vertices. Instead, we shift the partition simplex S_0 to $S'_0 = S(v_{10}, v_{11}, v'_{21})$ and construct Q' from the components S'_0 and S_1 as follows:

$$Q' = S'_0 \mho S_1 \equiv \begin{cases} S'_0 \cup S_1 & \text{if } \ (S'_0)^\circ \cap (S_1)^\circ = \emptyset \\ S'_0 \setminus S_1 & \text{if } \ S_1 \subset S'_0 \\ S_1 \setminus S'_0 & \text{if } \ S'_0 \subset S_1 \end{cases} \tag{43}$$

and otherwise the "union" operation $\mho$ is undefined. When v'_{21} lies on one of the lines H_j, Q' reduces to a triangle. We rule out such degeneracies. We say that v_{21} crosses H_j if v_{21} and v'_{21} lie on opposite sides of H_j. There are five cases to consider depending on which hyperplanes v_{21} crosses:

Case	$\mathcal{H}_I$ crossings	$\mathcal{H}_S$ crossings	Q'	Determinantal?
a)	0	0	endoskeletal	Yes
b)	0	1	non-convex	Yes
c)	1	0	non-convex	Yes
d)	1	1	not defined	No
e)	1	2	non-convex	Yes

We shall now explain the results in the last two columns, picking out Cases b), c) and d) for illustrative purposes. In Case a), when Q' is

endoskeletal,

$$\text{(44)} \qquad \begin{aligned} \operatorname{area} Q' &= \det(v_{11}-v_{10},\ v_{21}-v_{20})/2 \\ &= \det(v_{11}-v_{10}, v_{21}-v_{10})/2 + \det(v_{11}-v_{10}, v_{10}-v_{20})/2 \\ &= \operatorname{area} S_0 + \operatorname{area} S_1, \end{aligned}$$

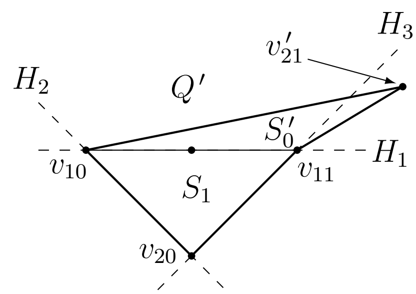
where all the determinants in (44) are positive for the given configuration. As v_{21} moves around, the determinant

$$D_0 \equiv \det(v_{11}-v_{10}, v_{21}-v_{10})$$

changes sign when v_{21} crosses the internal hyperplane H_1.

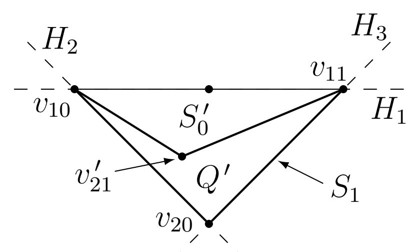
Case b): v_{21} crosses H_3 but neither H_1 nor H_2

In this case, $Q' = S_0' \cup S_1$ is a well-defined non-convex polyhedron. The determinants in (44) stay positive and the volume formula remains valid. The subcase when v_{21} crosses H_2 (but neither H_1 nor H_3) is exactly the same.



Case c): v_{21} crosses H_1 but neither H_2 nor H_3

v_{21}' lies below the line H_1 but inside the triangle S_1, so that S_0' lies inside S_1. The quadrilateral $Q' = S_1 \backslash S_0'$ is a well-defined non-convex polyhedron. The area formula (44) makes perfect sense for this Q': the determinant D_0 is now negative and so (44) reads



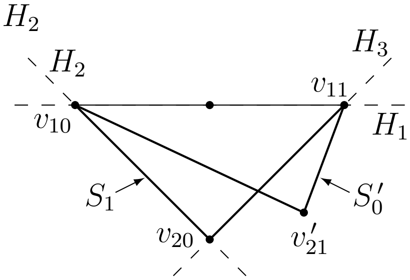
$$\operatorname{area} Q' = -\operatorname{area} S_0 + \operatorname{area} S_1,$$

i.e., the area contributions subtract, in accordance with the geometry. The determinant formula in its wisdom has taken proper account of the signs!

Case e) is similar except that the roles of S_0 and S_1 are reversed: v_{21}' lies below both H_2 and H_3; $S_1 \subset S_0'$; the resulting non-convex polyhedron $Q' = S_0 \backslash S_1'$ is well-defined and

$$\operatorname{area} Q' = \det(v_{11}-v_{10},\ v_{20}-v_{21}')/2.$$

Case d): v_{21} crosses H_1 and H_3 but not H_2



In this case Q' is not a well-defined polyhedron because S'_0 and S_1 overlap, with neither contained in the other. We reject this "butterfly" figure for which the area formula (44) has no sensible interpretation. Of course, with the vertices in this configuration we could reinterpret Q' as an endoskeletal polytope with bones $S(v_{10}\, v'_{21})$ and $S(v_{20}\, v_{11})$.

Guided by the above example, we can readily produce examples of non-convex determinantal polyhedra in higher dimensions. Let P be any polytope in $\mathcal{E}^d_\pi$ and consider its backbone partition (22) with backbone B_1. As we saw in (37),

$$\text{vol}\, P = \frac{1}{d!}|\det M(\mathcal{A})| = \frac{1}{d!}\sum_{\boldsymbol{l}\in\mathcal{L}} \sigma_{\boldsymbol{l}}\, D_{\boldsymbol{l}}, \tag{45}$$

where $\mathcal{A}$ is the set of axis vectors of P, $D_{\boldsymbol{l}} = \det M(\mathcal{V}(S_{\boldsymbol{l}}))$ and $\sigma_{\boldsymbol{l}} = \text{sign}\, D_{\boldsymbol{l}}$. Suppose we move around one of the vertices v of P. $D_{\boldsymbol{l}}$ changes sign if and only if $v \in \mathcal{V}(S_{\boldsymbol{l}})$ and v crosses a hyperplane $H \in \mathcal{H}_I$ which contains all the other vertices of $S_{\boldsymbol{l}}$. There is no sign change if v crosses a hyperplane in $\mathcal{H}_S$:

Example 4. *Non-convex determinantal polyhedra (after Example 3b)*

Let $P \in \mathcal{E}^d_\pi$ with $\dim B_2 = 1$. We take P in standard form and, as in Figure 4, we project P onto a "spotted kite" $\tilde{P}$ in the $x_1 x_s$-plane where $a_{11} = v_{11} - v_{10} = e_1$ and $a_{21} = v_{21} - v_{20} = e_s$:

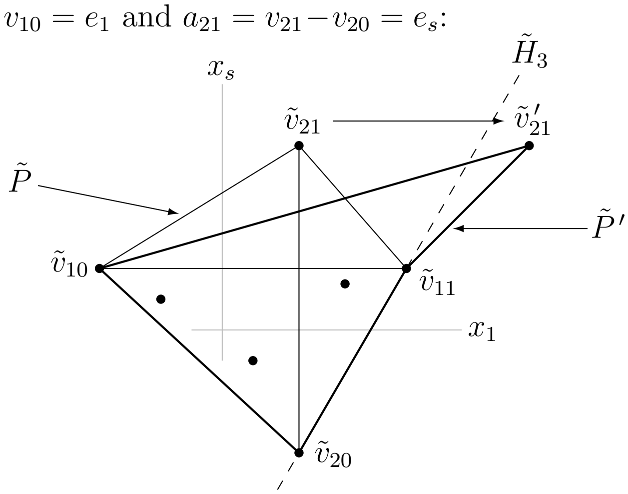


FIGURE 7. $\tilde{P}$ and $\tilde{P}'$ in the $x_1 x_s$-plane

Let $H_3 \in \mathcal{H}_S(P)$ be any supporting plane which contains the vertices v_{11} and v_{20}. In Figure 7 there is no point in drawing the projection of H_3 because it would fill the entire x_1x_s-plane. Rather, we have represented H_3 by the dashed line $\tilde{H}_3$ passing through the points $\tilde{v}_{20}$ and $\tilde{v}_{11}$.

The projected figure now looks like the quadrilateral Q of Example 3. As in Case b) of Example 3, we translate v_{21} in the direction e_1 to a point $v_{21}^{'}$ on the other side of H_3 and we drag along all the partition simplices with v_{21} as a vertex:

$$S_{\boldsymbol{l}} \to S_{\boldsymbol{l}}' = S(\mathcal{V}(S_{\boldsymbol{l}}) \cup \{v_{21}'\} \setminus \{v_{21}\}). \tag{46}$$

Each vertex set $\mathcal{V}(S_{\boldsymbol{l}})$ contains one vertex of B_2, either v_{20} or v_{21}. If it contains v_{20} then $S_{\boldsymbol{l}}' = S_{\boldsymbol{l}}$.

We claim that the simplices $S_{\boldsymbol{l}}'$, $\boldsymbol{l} \in \mathcal{L}$, are essentially disjoint. For consider two simplices, $S_{\boldsymbol{k}}'$ and $S_{\boldsymbol{l}}'$, $\boldsymbol{k} \neq \boldsymbol{l}$. Now v_{21} is in i) both of $\mathcal{V}(S_{\boldsymbol{k}})$ and $\mathcal{V}(S_{\boldsymbol{l}})$; ii) neither $\mathcal{V}(S_{\boldsymbol{k}})$ or $\mathcal{V}(S_{\boldsymbol{l}})$; or iii) one of $\mathcal{V}(S_{\boldsymbol{k}})$ and $\mathcal{V}(S_{\boldsymbol{l}})$. If i), then both simplices move together and their interiors remain disjoint. If ii), then both simplices remain fixed and are essentially disjoint. If iii), suppose that $v_{21} \in \mathcal{V}(S_{\boldsymbol{k}})$. There is some hyperplane Y which separates the convex sets $S_{\boldsymbol{k}}$ and $S_{\boldsymbol{l}}$ (see [3], p. 11) which must contain the bone B_1 with its axis e_1. Hence, when v_{21} is translated in the e_1 direction it moves parallel to Y and $S_{\boldsymbol{k}}'$ remains on its side of Y, essentially disjoint from $S_{\boldsymbol{l}}'$.

We define $P' = \bigcup_{\boldsymbol{l} \in \mathcal{L}} S_{\boldsymbol{l}}'$. This is our desired polyhedron. Clearly, P' is not convex because the line segment $S(v_{20}, v_{21}') \not\subset P'$. As for its volume, all the quantities in (45) are invariant under the translation $v_{21} \to v_{21}'$ and we conclude that P' is determinantal:

$$\operatorname{vol} P' = \frac{1}{d!} |\det M(\mathcal{A}')|, \quad \mathcal{A}' = \mathcal{A} \cup \{a_{21}'\} \setminus \{a_{21}\} \tag{47}$$

where $a_{21}' = v_{21}' - v_{20}$.

Example 5. *Non-convex determinantal polyhedra (after Example 3c)*

We start with the standard endoskeletal polytope $P \in \mathcal{T}_\pi^d$ constructed in Lemma 5 and form its backbone partition $P = \bigcup_{\boldsymbol{l} \in \mathcal{L}} S_{\boldsymbol{l}}$ with backbone B_1. Focussing on the bone B_2, we simplify notation by setting $m = d_2$ and $f_k = e_{d_1+k}$, $k = 1, \cdots, m$. Then $v_{20} = -(f_1 + \cdots + f_m)/(m+1)$ and $v_{2k} = v_{20} + f_k$, $k = 1, \cdots, m$. Consider the hyperplane

$$H = \{x \in \mathbb{R}^d \mid h(x) \equiv (2f_1 + f_2 + \cdots + f_m) \cdot x = 0\}.$$

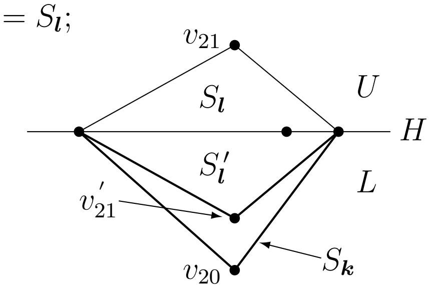
It's easy to check that $h(v_{jk}) = 0$ for $j \neq 2$, $h(v_{20}) = -1$, $h(v_{21}) = 1$ and $h(v_{2k}) = 0$ for $k > 1$. Thus all the vertices of P lie on H except for v_{20} and v_{21} which lie on opposite sides of H. We call the side U of

H containing v_{21} the "upper" side and the side L containing v_{20} the "lower" side.

We reposition v_{21} to $v'_{21} \in L \cap P$, say, on the line segment $S(v_{20}, v_{21})$. This shift produces a corresponding shift (46) in the partition simplices. For each $S_{\boldsymbol{l}}$ we distinguish three cases:

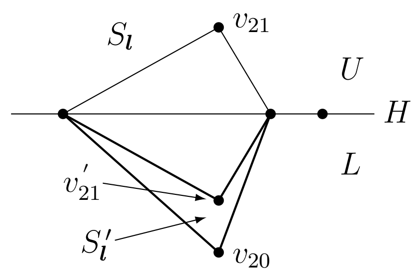
a) $l_2 = 1$: $v_{21} \notin \mathcal{V}(S_{\boldsymbol{l}})$, $S_{\boldsymbol{l}} \subset L$, and $S'_{\boldsymbol{l}} = S_{\boldsymbol{l}}$;

b) $l_2 = 0$: $v_{20} \notin \mathcal{V}(S_{\boldsymbol{l}})$ so that $S_{\boldsymbol{l}} \subset U$. Let $S_{\boldsymbol{k}}$ be the partition simplex obtained from $S_{\boldsymbol{l}}$ by replacing the vertex v_{21} by v_{20}. Then $S'_{\boldsymbol{l}} \subset S_{\boldsymbol{k}} \subset L$.



c) $l_2 \neq 0, 1$: $v_{20}, v_{21} \in \mathcal{V}(S_{\boldsymbol{l}})$.

In this case $S_{\boldsymbol{l}}$ extends across H. However, $S'_{\boldsymbol{l}} \subset L$ and $S'_{\boldsymbol{l}} \subset S_{\boldsymbol{l}}$ because we have placed v'_{21} inside $S_{\boldsymbol{l}}$.



We would like to define the shifted polyhedron P' as the union of the shifted partition simplices $\{S'_{\boldsymbol{l}}\}$. However, the $S'_{\boldsymbol{l}}$ are not disjoint, as is evident from Case b) above. (In this discussion, we shall use the word "disjoint" as a short form for "essentially disjoint", where A and B are essentially disjoint if $A^\circ \cap B^\circ = \emptyset$.) To deal with this complication we partition the index set $\mathcal{L}$ of (35) as $\mathcal{L} = \mathcal{L}_a \cup \mathcal{L}_b \cup \mathcal{L}_c$, where

$$\mathcal{L}_a = \{l \in \mathcal{L} \mid l_2 = 1\}, \quad \mathcal{L}_b = \{l \in \mathcal{L} \mid l_2 = 0\}, \quad \mathcal{L}_c = \{l \in \mathcal{L} \mid l_2 \neq 0, 1\},$$

and we set

$$P_a = \bigcup_{l \in \mathcal{L}_a} S_{\boldsymbol{l}}, \quad P_b = \bigcup_{l \in \mathcal{L}_b} S'_{\boldsymbol{l}}, \quad P_c = \bigcup_{l \in \mathcal{L}_c} S'_{\boldsymbol{l}}.$$

It is easy to see that the constituent simplices in each of these unions are disjoint and that $P_b \subset P_a$. We now define

$$P' = P_a \backslash P_b \cup P_c. \tag{48}$$

Note that the union in (48) is disjoint because for any $S'_{\boldsymbol{l}}$, $l \in \mathcal{L}_c$, we have $S'_{\boldsymbol{l}} \subset S_{\boldsymbol{l}}$ which is disjoint from P_a. It follows from (48) that

$$\operatorname{vol} P' = \operatorname{vol} P_a - \operatorname{vol} P_b + \operatorname{vol} P_c\,. \tag{49}$$

We claim that the determinant formula (45) respects the algebra of (49). Consider the sign changes in (45) when we relocate $v_{21} \to v'_{21}$ while holding the $\sigma_{\boldsymbol{l}}$'s fixed at their initial values. For $l \in \mathcal{L}_a$, $D_{\boldsymbol{l}}$ is unchanged, whereas for $l \in \mathcal{L}_b$, $D_{\boldsymbol{l}}$ changes sign. As for $l \in \mathcal{L}_c$, we claim that $D_{\boldsymbol{l}}$ does not change sign. To see this, we regard $D_{\boldsymbol{l}}$ as a (linear) function of v_{21} and we consider the function $f(t) \equiv D_{\boldsymbol{l}}(v_{21}(t))$ as $v_{21}(t)$ moves along the line $(1-t)v_{21} + tv_{20}$, $0 \le t \le 1$. Since $f(t)$ is a linear function with $f(1) = 0$, it does not change sign. Thus, with the shifted axes $\mathcal{A}'$ as in (47), we have

$$|\det M(\mathcal{A}')| = \sum_{l \in \mathcal{L}_a} |D_{\boldsymbol{l}}| - \sum_{l \in \mathcal{L}_b} |D_{\boldsymbol{l}}| + \sum_{l \in \mathcal{L}_c} |D_{\boldsymbol{l}}|. \tag{50}$$

(49) and (50) yield the desired conclusion (47) and we obtain our nonconvex determinantal polyhedron P'.

Finally, we complete the discussion of Example 1:

Example 1 (continued) The shifted octahedron $\hat{\mathcal{O}}^3$

Let T be the tetrahedron $S(v_{10}, v_{20}, v_{30}, v_{31})$ and let P' be the nonconvex polyhedron $P' = \hat{\mathcal{O}}^3 \setminus T$. As we explained following Figure 7, (30) correctly gives the volume of P', i.e., $\operatorname{vol} P' = \frac{1}{3!}\det(a_{11},\, a_{21},\, a_{31})$, where the determinant is positive for the pictured configuration. Thus, $3!\operatorname{vol}\hat{\mathcal{O}}^3 = 3!(\operatorname{vol} P' + \operatorname{vol} T)$ is given by

$$\det(v_{11}-v_{10},\, v_{21}-v_{20},\, v_{31}-v_{30}) + \det(v_{30}-v_{10},\, v_{20}-v_{10},\, v_{31}-v_{10}),$$

by (2). Our assertion before Figure 3 is now clear: this expression cannot be rewritten as a single determinant whose columns are differences of vertex vectors of $\hat{\mathcal{O}}^3$. The sobering moral of this example is that not all octahedra are endoskeletal or determinantal.

It may be possible to carry out a more systematic analysis of nonconvex polyhedra, but as such an analysis was not part of the original mandate of this paper we content ourselves with the above examples.

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References

[1] N. Arkani-Hamed and J. Trnka, The Amplituhedron, J. High Energy Phys. (2014) 30

[2] J. Goodman and J. O'Rourke, Handbook of Discrete and Computational Geometry, 2nd Ed., CRC, Boca Raton, 2004

[3] B. Grünbaum, Convex Polytopes, 2nd Ed., Springer-Verlag, New York, 2003

[4] W. Kindree and L. Rosen, A Class of Polytopes with a Simple Volume Formula